Large Characteristic Subgroups in Infinite Groups

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Michio Suzuki

October 2, 1926 – May 31, 1998

Michio Suzuki

Michio Suzuki in 1962

Lemma 21.1.4 of the book by **Kargapolov** and **Merzlyakov** 1971

Let G be a group containing an abelian subgroup of finite index. Then G has an abelian characteristic subgroup of finite index.

Mikhail I. Kargapolov (1928–1976)   Yuriĭ I. Merzlyakov (1940–1995)

Proof

Proof

Let A be an abelian subgroup of finite index of G, and let A^* be the smallest characteristic subgroup of G containing A. Then there exist finitely many automorphisms $\theta_1, \ldots, \theta_n$ of G such that

$$A^* = \langle A^{\theta_1}, \ldots, A^{\theta_n} \rangle.$$

It follows that

$$A^{\theta_1} \cap \ldots \cap A^{\theta_n}$$

is contained in the centre of A^*, and so $Z(A^*)$ is an abelian characteristic subgroup of finite index of G. □

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Let G be a group containing an abelian subgroup A of finite index. Then G has an abelian characteristic subgroup B of finite index such that $[A, B] = \{1\}$.

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Let G be a group containing an abelian subgroup A of finite index. Then G has an abelian characteristic subgroup B of finite index such that $[A, B] = \{1\}$.

Moreover, if $|G : A| = n$, the subgroup B can be chosen of index at most n^n.

Theorem 1.41 of Isaacs' book

Let G be a finite group containing an abelian subgroup of index n. Then G has an abelian characteristic subgroup of index at most n^2.

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Francesco **de Giovanni** and *Marco* **Trombetti**

Comm. Algebra 2018

Let G be a group containing an abelian subgroup of finite index n. Then G has an abelian characteristic subgroup of index at most n^2.

Brunella **Bruno** and *Franco* **Napolitani**

Glasgow Math. J. 2004

Let G be a group containing a subgroup of finite index which is nilpotent of class at most k.
Then G has also a characteristic subgroup of finite index which is nilpotent of class at most k.

Evgeny **Khukhro** and *Natalia* **Makarenko**

J. London Math. Soc. 2018

Let G be a group containing a subgroup of finite index which is soluble of derived length at most k.
Then G has also a characteristic subgroup of finite index which is soluble of derived length at most k.

A group class $\mathfrak{X}$ is said to be *F-characteristic*
if any group containing an $\mathfrak{X}$-subgroup of finite index
has also a characteristic subgroup of finite index in the class $\mathfrak{X}$

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Each of the following classes is F-characteristic:

- the class $\mathfrak{A}$ of all abelian groups
- the class $\mathfrak{N}$ of all nilpotent groups
- the class $\mathfrak{N}_k$ of all nilpotent groups of class at most k
- the class $\mathfrak{S}$ of all soluble groups
- the class $\mathfrak{S}_k$ of all soluble groups of derived length at most k

Some group classes which are not *F-characteristic*

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$$G = F \times \langle a \rangle$$

where F is free of countably infinite rank and a has prime order

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  Swan's theorem: every torsion-free group containing a free subgroup of finite index is free

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Some group classes which are not *F-characteristic*

- the class of all free groups
- the class of all torsion-free groups
- the class of all free abelian groups
  The only characteristic subgroups of a free abelian group F are the powers F^n, with $n \geqslant 0$

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- the class of all free abelian groups
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$$G = Alt(5) \times Alt(5)$$

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A group G is called *quasihamiltonian* if $XY = YX$ for all subgroups X and Y of G.

A G T A

Two further *F-characteristic* group classes

- the class of all quasihamiltonian groups
- the class of all groups with modular subgroup lattice

A group G is called *quasihamiltonian* if $XY = YX$ for all subgroups X and Y of G. It is known that a group is quasihamiltonian if and only it is locally nilpotent and has a modular subgroup lattice

The class of T-groups

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The structure of soluble T-groups has been described by W. Gaschütz (1957) and D.J.S. Robinson (1964)

Some relevant properties of soluble T-groups

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  If G is a soluble T-group, every subgroup of G' is normal in G, and so G acts on G' as a group of power automorphisms

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Some relevant properties of soluble T-groups

- Soluble T-groups are metabelian
- A finitely generated soluble T-group is either finite or abelian
- Any finitely generated soluble T-group is a $\overline{T}$-group, i.e. all its subgroups have the T-property
- If G is any soluble T-group, all its subgroups containing $Fit(G)$ are characteristic

Francesco **de Giovanni** and *Marco* **Trombetti**

Theorem

The class of periodic soluble T-groups is F-characteristic

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Theorem
The class of periodic soluble T-groups is F-characteristic

Corollary
The class of locally soluble $\overline{T}$-groups is F-characteristic

Non-periodic T-groups

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- G is of *type* 1 if the centralizer $C_G(G')$ is not periodic
  In this case it turns out that G is abelian-by-finite

Non-periodic T-groups

Let G be a soluble non-abelian T-group. Then:

- G is of *type* 1 if the centralizer $C_G(G')$ is not periodic
- G is of *type* 2 if $C_G(G')$ is periodic but G contains elements of infinite order

Non-periodic T-groups

Let G be a soluble non-abelian T-group. Then:

- G is of *type* 1 if the centralizer $C_G(G')$ is not periodic
- G is of *type* 2 if $C_G(G')$ is periodic but G contains elements of infinite order

  There exists a metabelian group containing a subgroup of finite index which is a T-group of type 2 but no characteristic subgroup of finite index with the T-property

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- If G is any group with a soluble subgroup X of finite index which is a T-group of type 2, then X' is characteristic in G

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- The set of all elements of finite order is a subgroup
- The commutator subgroup is divisible
- If X is a subgroup of finite index of a soluble T-group of type 2, then X is also a T-group of type 2, and $X' = G'$
- If G is any group with a soluble subgroup X of finite index which is a T-group of type 2, then X' is characteristic in G. Moreover, if T is the subgroup of all elements of finite order of X, also T' is characteristic in G

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Theorem

Let G be a group containing a soluble subgroup X of finite index which is a T-group of type 2. If either X has finite torsion-free rank or X' has finite sectional rank, then G has a characteristic subgroup of finite index which is a T-group (of type 2)

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Corollary

The class of soluble T-groups of finite torsion-free rank is F-characteristic

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